Midterm 2 Review Questions

There will be seven multiple-choice questions and three work-out problems in Midterm 2.

For a review of material about the concepts covered in Midterm 2, please refer to the pdf named “Formula sheet Midterm 2” in the content folder. Below are some selected questions from the past final exams. You can find more exercises and answers from the past exam archive.

1. Spring 2019: #13.
2. Spring 2019: #16.
3. Spring 2019: #18.
4. Spring 2018: #11.
5. Spring 2018: #13.
6. Spring 2018: #20.
   - Related question: Fall 2018 #18 from MA266
7. Spring 2017: #3.
8. Fall 2019: #14.
9. Fall 2018: #13.
10. Fall 2012: #23.

We list the above questions here for convenience. The complete notes for solving these questions will be posted on Thursday, March 31.

1. Spring 2019: #13.

   Keywords: eigenvalue, invertible, diagonalizable.

   Which of the following statements are true?
   (i) If λ is an eigenvalue for A, then $-\lambda$ is an eigenvalue for $-A$.
   (ii) If zero is an eigenvalue of A, then A is not invertible.
   (iii) If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.
   (iv) Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then A is both invertible and diagonalizable.
   A. (i) and (ii) only
   B. (i) and (iii) only
   C. (i), (ii) and (iii) only
   D. (i), (ii) and(iv) only
   E. (i), (ii), (iii) and (iv)

   (i) True. $A\vec{v} = \lambda \vec{v} \Rightarrow -A\vec{v} = -\lambda \vec{v}$

   In general, $A\vec{v} = \lambda\vec{v} \Rightarrow kA\vec{v} = k\lambda\vec{v}$

   So $k\lambda$ is an eigenvalue for kA for any $k \in \mathbb{R}$.

(2) True by the IMT.

(3) Not True. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(4) Not True.

A is invertible since $\det A \neq 0$

A is not diagonalizable.

- A has only one eigenvalue $\lambda = 2$ (twice).
- Eigenvector(s)?

$(A-2I)\vec{v} = \vec{0}$.

$\Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_2 = 0$

$\Rightarrow$ Any eigenvector for $\lambda = 2$ must be a scalar multiple of $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

In general, any matrix $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is not diagonalizable.

2. Spring 2019: #16.

Keywords: graph of $\mathbf{x}' = A\mathbf{x}$

Consider the dynamical system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then the origin is

A. an attractor
B. a repeller
C. a saddle point
D. a spiral point
E. none of the above

It sufficies to check the eigenvalues for A:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 2 & 0-\lambda \end{vmatrix} = \lambda(\lambda - 1) - 2 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 2 \text{ and } \lambda = -1$$

Trajectories for the System $\mathbf{x}' = A\mathbf{x}$:

- **attractor**: A has distinct negative real eigenvalues.
- **repeller**: A has distinct positive real eigenvalues.
- **saddle point**: A has real eigenvalues of opposite sign.
- **spiral point**: A has complex conjugate eigenvalues with nonzero real parts.
- **center**: A has purely imaginary eigenvalues.

3. Spring 2019: #18.

Keywords: **subspaces of $\mathbb{R}^n$**

Related questions: Fall 2018 #10, Fall 2017 #8, Fall 2012 #7.

Which of the following subsets of the vector space $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$?

(i) The set of all vectors $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with the property $2xyz = 0$.

(ii) The set of all the solutions of the equation $x - 5y + 2z = 0$.

(iii) The set of all solutions for the system $A = \begin{bmatrix} 2 & 3 & 2 \\ 5 & 2 & 8 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(iv) The set of all the solutions of the equation $x + 3y = 2z + 1$.

A. (ii) and (iii) only

B. (ii) and (iv) only

C. (iii) and (iv) only

D. (ii), (iii) and (iv) only

E. (i), (ii), (iii) and (iv)

(1) Not.

$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are in the subset

but $\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the subset.

(2) Yes. $x - 5y + 2z = 0$ is a plane through the origin.

(3) Yes. It is Nul A.

(4) Not. Since $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the set. In fact, it is a plane not passing the origin.

4. Spring 2018: #11.

Keywords: subspaces of $\mathbb{P}_n$ or $\mathbb{M}_{n\times n}$

Related questions: Fall 2015 #5, Fall 2015 #6, Spring 2017 #19, Spring 2007 #8, Fall 2019#9

Let P_3 be the real vector space of polynomials of degree less than or equal to 3, together with the zero polynomial. Let W be the subspace of P_3 consisting of all the polynomials $p(t) \in P_3$ satisfying $p(0) = p(1)$. What is the dimension of W?

A. 0
B. 1
C. 2
D. 3
E. 4

- A general element in $\mathbb{P}_3$ can be written as

$$p(t) = at^3 + bt^2 + ct + d$$

- W is the subspace s.t.

$$p(0) = p(1)$$

$$\Rightarrow a\cdot 0 + b\cdot 0 + c\cdot 0 + \cancel{d} = a\cdot 1 + b\cdot 1 + c\cdot 1 + \cancel{d}$$

$$\Rightarrow a + b + c = 0$$

$$\Rightarrow a = -b - c$$

- Thus any $p(t) \in W$ has to be in the form

$$p(t) = (-b-c)t^3 + \underline{b}t^2 + \underline{c}t + \underline{d}$$

i.e. $p(t) \in W$ is determined by the values b, c, d.
Thus it is 3-dim'l.

In fact, $p(t) = b(-t^3+t^2) + c(-t^3+t) + d\times 1 \in W$.

So any $p(t) \in W$ is spanned by $\{-t^3+t^2, -t^3+t, 1\}$.
which is a linearly independent set. and contains 3 elements. So $\dim W = 3$.

5. Spring 2018: #13.

Keywords: Rank Theorem, Row A, Nul A, Col A

Let A be a 5×7 real matrix such that the dimension of its column space is equal to 5. Which of the following statements is true?

A. The dimension of the null space of A is equal to 0 ✗

B. The columns of A are linearly independent ✗

C. The rows of A are linearly independent ✓

D. The rank of A^T is equal to 7 ✗ $\text{rank } A^T = \text{rank } A$

E. The dimension of the row space of A is 2 ✗

C. True.

$$\dim \text{Row } A = \dim \text{Col } A = \text{Rank } A = 5$$

- $\dim \text{Col } A = \text{rank } A = 5$

- Rank Thm: $\text{rank } A + \text{nullity } A = \#\text{columns of } A$

$$5 + \underset{2}{\underset{\parallel}{}} = 7$$

$\Rightarrow \text{nullity } A = \dim \text{Nul } A = 2$.

6. (1) Spring 2018: #20.

Keywords: Solutions to $\mathbf{x}' = A\mathbf{x}$

Related question: Fall 2018 #18 from MA266

Let $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be the solution of the following initial value problem:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

Then $y(1)$ is equal to

A. $3e^6$

B. $-3e^6$

C. $3e^2$

D. $-3e^2$

E. $3e^6 - 3e^2$

: Eigenvalue

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (\lambda - 4)^2 - 4 = 0$$

$$\Rightarrow \lambda - 4 = \pm 2 \Rightarrow \lambda_1 = 2 \text{ and } \lambda_2 = 6$$

· Eigenvectors : For $\lambda_1 = 2$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

Thus $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 2$

For $\lambda_2 = 6$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

Thus $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 6$.

Thus the general solution is

$$\vec{x}(t) = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

The initial condition

$$\vec{x}(0) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 3 \\ -c_1 + c_2 = -3 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = 0 \end{cases}$$

Thus

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

We need to compute $y(1)$.

Since $y(t) = -3e^{2t}$

$$y(1) = -3e^{2}$$

6. (2) Fall 2018 #18 from MA266

The real 2×2 matrix A has an eigenvalue $\lambda_1 = 3 + 2i$ with the corresponding eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$.

Then the general solution of the system of differential equations

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

is $\mathbf{x}(t) =$

A. $C_1 e^{3t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix}$

B. $C_1 e^{3t} \begin{pmatrix} -2\cos 2t \\ \sin 2t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \sin 2t \\ -2\cos 2t \end{pmatrix}$

C. $C_1 e^{3t} \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$

D. $C_1 e^{3t} \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix}$

E. $C_1 e^{3t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$

From the given information,

We know.

$\vec{x}(t) = \vec{V}_1 e^{\lambda_1 t}$ is a solution.

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(3+2i)t} \qquad \left(e^{(x+iy)t} = e^{xt}(\cos yt + i\sin yt) \right)$$

$$= \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{3t}(\cos 2t + i\sin 2t)$$

$$= e^{3t} \begin{bmatrix} \cos 2t + \underline{i\sin 2t} \\ \underline{2i\cos 2t} - 2\sin 2t \end{bmatrix}$$

$$= \underbrace{e^{3t} \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix}}_{\vec{x}_1(t)} + i \underbrace{e^{3t} \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}}_{\vec{x}_2(t)}$$

7. Spring 2017: #3.

Keywords: singular, similar, solution to $A\mathbf{x} = \mathbf{b}$

Let A, B be two 4×4 matrices. Which of the following statements is ALWAYS true?
A. If $A + B$ is singular, then either A or B is singular.
B. If AB is symmetric, then BA is also symmetric.
C. If A and B are similar matrices, then $\det(A) = \det(B)$.
D. If $AB = AC$, then $B = C$.
E. If $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b} \neq \mathbf{0}$, then A is non-singular.

A. False

Recall singular $\Longleftrightarrow$ not invertible

$A = I_4$, $B = -A = -I_4$

B. False

Recall A is symmetric $\Longleftrightarrow$ $A^T = A$

So $(AB)^T = AB$

$\parallel$

$B^T A^T$

C. True.

D. False

E. False.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

8. Fall 2019: #14.

Keywords: subspace of $\mathbb{M}_{n\times n}$

Let $\mathbb{M}_{3\times 3}$ be the space of 3×3 matrices. Let H be the subspace of $\mathbb{M}_{3\times 3}$ consisting of all matrices $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ such that $A\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. What is the dimension of H ?

A. 1
B. 3
C. 6
D. 8
E. 9

$$\Downarrow$$

$$\begin{bmatrix} a-2b \\ d-2e \\ g-2h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a=2b \\ d=2e \\ g=2h \end{cases}$$ Thus $A\in H$ has the form $\begin{bmatrix} 2b & \underline{b} & \underline{c} \\ 2e & \underline{e} & \underline{f} \\ 2h & \underline{h} & \underline{i} \end{bmatrix}$

i.e. any $A\in H$ is determined by the 6 constants b, c, e, f, h, i.

So H is 6 dim'l.

In fact, $\begin{bmatrix} 2b & b & c \\ 2e & e & f \\ 2h & h & i \end{bmatrix} = b\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$+ e\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$+ h\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So any element in H is spanned by the 6 matrices appears on the RHS. They are also linearly independent. These are the basis for H.

9. Fall 2018: #13.

Keywords: eigenvalue

Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 5 & -2 & -2 \\ -2 & 3 & 3 & 5 \end{bmatrix}$. Which of the following values is a multiple eigenvalue of A?

A. -1
B. -2
C. 1
D. 2
E. 4

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 1 & -1-\lambda & 0 & 0 \\ 3 & 5 & -2-\lambda & -2 \\ -2 & 3 & 3 & 5-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} -1-\lambda & 0 & 0 \\ 5 & -2-\lambda & -2 \\ 3 & 3 & 5-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-1-\lambda) \begin{vmatrix} -2-\lambda & -2 \\ 3 & 5-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-1-\lambda) \underbrace{\left[(\lambda+2)(\lambda-5) + 6 \right]}_{\begin{array}{c} \shortparallel \\ \lambda^2 - 3\lambda - 10 + 6 \\ = \lambda^2 - 3\lambda - 4 \\ = (\lambda-4)(\lambda+1) \end{array}} = 0$$

$$\Rightarrow (2-\lambda)(+1+\lambda)(\lambda-4)(\lambda+1) = 0$$

$\Rightarrow \lambda = -1$ has multiplicity 2.

10. Fall 2012: #23.

Keywords: diagonalization

Let $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$. For which matrix P is it true that $P^{-1}AP = D$, where D is a diagonal matrix?

A. $P = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$

B. $P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$

C. $P = \begin{bmatrix} -1 & 1 \\ 4 & 1 \end{bmatrix}$

D. $P = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$

E. $P = \begin{bmatrix} 1/3 & 2/3 \\ 1/6 & -1/6 \end{bmatrix}$

Recall diagonalization of a matrix A:

· Find eigenvalues

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-1) - 4$$

$$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0$$

$\Rightarrow \lambda = -3$ and $\lambda = 2$.

· Find eigenvectors:

For $\lambda = -3$. $\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 2$. $\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right]$

$\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

We need to choose P such that its two columns are at least a scalar multiple of the $\vec{v}_1$ and $\vec{v}_2$ we found

· We can take $P = [\vec{v}_1 \ \ \vec{v}_2] = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$

and $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.